# 2 行列式論

### 2.10.1 行列式の交代多重線形性

## C

(1) 行列式の交代多重線形性を用いて, 複素数を要素とする次の行列の行列式を求めよ.

**[(1-3) の解答例]**

$$\det\begin{pmatrix} a & b & c & d \\ d & a & b & c \\ c & d & a & b \\ b & c & d & a \end{pmatrix} = \det\begin{pmatrix} a+b+c+d & b & c & d \\ a+b+c+d & a & b & c \\ a+b+c+d & d & a & b \\ a+b+c+d & c & d & a \end{pmatrix} \quad \left(\begin{array}{l}\text{第 1 列に第 2 列, 第 3 列,} \\ \text{第 4 列をそれぞれ足す.}\end{array}\right)$$

$$= (a+b+c+d)\cdot\det\begin{pmatrix} 1 & b & c & d \\ 1 & a & b & c \\ 1 & d & a & b \\ 1 & c & d & a \end{pmatrix} \quad \left(\begin{array}{l}\text{第 1 列から } (a+b+c+d) \\ \text{をくくり出す.}\end{array}\right)$$

$$= (a+b+c+d)\cdot\det\left(\begin{array}{c|ccc} 1 & b & c & d \\ 0 & a-b & b-c & c-d \\ 0 & -(b-d) & a-c & b-d \\ 0 & c-b & d-c & a-d \end{array}\right) \quad \left(\begin{array}{l}\text{第 2 行, 第 3 行, 第 4 行から} \\ \text{第 1 行をそれぞれ引く.}\end{array}\right)$$

$$= (a+b+c+d)\cdot(-1)^{1+1}\cdot\det\begin{pmatrix} a-b & b-c & c-d \\ -(b-d) & a-c & b-d \\ c-b & d-c & a-d \end{pmatrix} \quad \left(\begin{array}{l}\text{第 1 列で} \\ \text{ラプラス展開.}\end{array}\right)$$

$$= (a+b+c+d)\cdot\det\begin{pmatrix} a-b+c-d & b-c & c-d \\ 0 & a-c & b-d \\ a-b+c-d & d-c & a-d \end{pmatrix} \quad \left(\text{第 1 列に第 3 列を足す.}\right)$$

$$= (a+b+c+d)(a-b+c-d)\cdot\det\begin{pmatrix} 1 & b-c & c-d \\ 0 & a-c & b-d \\ 1 & d-c & a-d \end{pmatrix} \quad \left(\begin{array}{l}\text{第 1 列から} \\ (a-b+c-d) \\ \text{をくくり出す.}\end{array}\right)$$

$$= (a+b+c+d)(a-b+c-d)\cdot\det\left(\begin{array}{c|cc} 1 & b-c & c-d \\ 0 & a-c & b-d \\ 0 & -(b-d) & a-c \end{array}\right) \quad \left(\begin{array}{l}\text{第 3 列から} \\ \text{第 1 行を引く.}\end{array}\right)$$

$$= (a+b+c+d)(a-b+c-d)\cdot(-1)^{1+1}\cdot\det\begin{pmatrix} a-c & b-d \\ -(b-d) & a-c \end{pmatrix} \quad \left(\begin{array}{l}\text{第 1 列で} \\ \text{ラプラス展開.}\end{array}\right)$$

$$= (a+b+c+d)(a-b+c-d)\left((a-c)^2+(b-d)^2\right). \quad \blacksquare$$

 

**[(1-4) の解答例]**

$$\det\begin{pmatrix} a & b & c & d & e \\ b & a & b & c & d \\ c & b & a & b & c \\ d & c & b & a & b \\ e & d & c & b & a \end{pmatrix} = \det\begin{pmatrix} a-e & b-d & c & d & e \\ b-d & a-c & b & c & d \\ 0 & 0 & a & b & c \\ -(b-d) & -(a-c) & b & a & b \\ -(a-e) & -(b-d) & c & b & a \end{pmatrix}$$

(第 1 列から第 5 列を引き, 第 2 列から第 4 列を引く.)

$$= \det\left(\begin{array}{cc|ccc} a-e & b-d & c & d & e \\ b-d & a-c & b & c & d \\ \hline 0 & 0 & a & b & c \\ 0 & 0 & 2b & a+c & b+d \\ 0 & 0 & 2c & b+d & a+e \end{array}\right)$$

(第 4 行に第 2 行を足し, 第 5 行に第 1 行を足す.)

$$= \det\begin{pmatrix} a-e & b-d \\ b-d & a-c \end{pmatrix} \cdot \det\begin{pmatrix} a & b & c \\ 2b & a+c & b+d \\ 2c & b+d & a+e \end{pmatrix}$$

$$= \big((a-e)(a-c)-(b-d)^2\big) \times \big(a(a+c)(a+e)+4bc(b+d)-2c^2(a+c)-a(b+d)^2-2b^2(a+e)\big). \quad \blacksquare$$

**[(1-5) の解答例]**

$$\det\begin{pmatrix} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 & x_1^5 \\ 1 & x_2 & x_2^2 & x_2^3 & x_2^4 & x_2^5 \\ 1 & x_3 & x_3^2 & x_3^3 & x_3^4 & x_3^5 \\ 1 & x_4 & x_4^2 & x_4^3 & x_4^4 & x_4^5 \\ 1 & x_5 & x_5^2 & x_5^3 & x_5^4 & x_5^5 \\ 1 & x_6 & x_6^2 & x_6^3 & x_6^4 & x_6^5 \end{pmatrix} = \det\left(\begin{array}{c|ccccc} 1 & x_1 & x_1^2 & x_1^3 & x_1^4 & x_1^5 \\ 0 & x_2-x_1 & x_2^2-x_1^2 & x_2^3-x_1^3 & x_2^4-x_1^4 & x_2^5-x_1^5 \\ 0 & x_3-x_1 & x_3^2-x_1^2 & x_3^3-x_1^3 & x_3^4-x_1^4 & x_3^5-x_1^5 \\ 0 & x_4-x_1 & x_4^2-x_1^2 & x_4^3-x_1^3 & x_4^4-x_1^4 & x_4^5-x_1^5 \\ 0 & x_5-x_1 & x_5^2-x_1^2 & x_5^3-x_1^3 & x_5^4-x_1^4 & x_5^5-x_1^5 \\ 0 & x_6-x_1 & x_6^2-x_1^2 & x_6^3-x_1^3 & x_6^4-x_1^4 & x_6^5-x_1^5 \end{array}\right)$$

(第 2 行, 第 3 行, 第 4 行, 第 5 行, 第 6 行から第 1 行をそれぞれ引く.)

$$= \det\begin{pmatrix} \underline{x_2-x_1} & x_2^2-x_1^2 & x_2^3-x_1^3 & x_2^4-x_1^4 & x_2^5-x_1^5 \\ \underline{x_3-x_1} & x_3^2-x_1^2 & x_3^3-x_1^3 & x_3^4-x_1^4 & x_3^5-x_1^5 \\ \underline{x_4-x_1} & x_4^2-x_1^2 & x_4^3-x_1^3 & x_4^4-x_1^4 & x_4^5-x_1^5 \\ \underline{x_5-x_1} & x_5^2-x_1^2 & x_5^3-x_1^3 & x_5^4-x_1^4 & x_5^5-x_1^5 \\ \underline{x_6-x_1} & x_6^2-x_1^2 & x_6^3-x_1^3 & x_6^4-x_1^4 & x_6^5-x_1^5 \end{pmatrix}$$

(第 1 列でラプラス展開.)

$$
\begin{aligned}
= \quad & (x_2-x_1)(x_3-x_1)(x_4-x_1)(x_5-x_1)(x_6-x_1) \\
& \times \det \begin{pmatrix}
1 & x_2+x_1 & x_2^2+x_2x_1+x_1^2 & x_2^3+x_2^2x_1+x_2x_1^2+x_1^3 & x_2^4+x_2^3x_1+x_2^2x_1^2+x_2x_1^3+x_1^4 \\
1 & x_3+x_1 & x_3^2+x_3x_1+x_1^2 & x_3^3+x_3^2x_1+x_3x_1^2+x_1^3 & x_3^4+x_3^3x_1+x_3^2x_1^2+x_3x_1^3+x_1^4 \\
1 & x_4+x_1 & x_4^2+x_4x_1+x_1^2 & x_4^3+x_4^2x_1+x_4x_1^2+x_1^3 & x_4^4+x_4^3x_1+x_4^2x_1^2+x_4x_1^3+x_1^4 \\
1 & x_5+x_1 & x_5^2+x_5x_1+x_1^2 & x_5^3+x_5^2x_1+x_5x_1^2+x_1^3 & x_5^4+x_5^3x_1+x_5^2x_1^2+x_5x_1^3+x_1^4 \\
1 & x_6+x_1 & x_6^2+x_6x_1+x_1^2 & x_6^3+x_6^2x_1+x_6x_1^2+x_1^3 & x_6^4+x_6^3x_1+x_6^2x_1^2+x_6x_1^3+x_1^4
\end{pmatrix}
\end{aligned}
$$

( 第 1 行から $(x_2-x_1)$, 第 2 行から $(x_3-x_1)$, 第 3 行から $(x_4-x_1)$,
第 4 行から $(x_5-x_1)$, 第 5 行から $(x_6-x_1)$ をそれぞれくくり出す. )

<u>以下、上記の手順を繰り返す.</u>

$$
\begin{aligned}
= \quad & (x_2-x_1)(x_3-x_1)(x_4-x_1)(x_5-x_1)(x_6-x_1) \\
& \times \det \left(\begin{array}{c|cccc}
1 & x_2+x_1 & x_2^2+x_2x_1+x_1^2 & x_2^3+x_2^2x_1+x_2x_1^2+x_1^3 & x_2^4+x_2^3x_1+x_2^2x_1^2+x_2x_1^3+x_1^4 \\
0 & x_3-x_2 & (x_3-x_2)(x_3+x_2+x_1) & (x_3-x_2)(x_3^2+x_2^2+x_1^2+x_1x_2+x_1x_3+x_2x_3) & (x_3-x_2)\left((x_3^2+x_2^2+x_1^2)(x_1+x_2+x_3)+x_1x_2x_3\right) \\
0 & x_4-x_2 & (x_4-x_2)(x_4+x_2+x_1) & (x_4-x_2)(x_4^2+x_2^2+x_1^2+x_1x_2+x_1x_4+x_2x_4) & (x_4-x_2)\left((x_4^2+x_2^2+x_1^2)(x_1+x_2+x_4)+x_1x_2x_4\right) \\
0 & x_5-x_2 & (x_5-x_2)(x_5+x_2+x_1) & (x_5-x_2)(x_5^2+x_2^2+x_1^2+x_1x_2+x_1x_5+x_2x_5) & (x_5-x_2)\left((x_5^2+x_2^2+x_1^2)(x_1+x_2+x_5)+x_1x_2x_5\right) \\
0 & x_6-x_2 & (x_6-x_2)(x_6+x_2+x_1) & (x_6-x_2)(x_6^2+x_2^2+x_1^2+x_1x_2+x_1x_6+x_2x_6) & (x_6-x_2)\left((x_6^2+x_2^2+x_1^2)(x_1+x_2+x_6)+x_1x_2x_6\right)
\end{array}\right)
\end{aligned}
$$

$$
\begin{aligned}
= \quad & (x_2-x_1)(x_3-x_1)(x_4-x_1)(x_5-x_1)(x_6-x_1) \\
& \times \det \begin{pmatrix}
\underline{x_3-x_2} & (x_3-x_2)(x_3+x_2+x_1) & (x_3-x_2)(x_3^2+x_2^2+x_1^2+x_1x_2+x_1x_3+x_2x_3) & (x_3-x_2)\left((x_3^2+x_2^2+x_1^2)(x_1+x_2+x_3)+x_1x_2x_3\right) \\
\underline{x_4-x_2} & (x_4-x_2)(x_4+x_2+x_1) & (x_4-x_2)(x_4^2+x_2^2+x_1^2+x_1x_2+x_1x_4+x_2x_4) & (x_4-x_2)\left((x_4^2+x_2^2+x_1^2)(x_1+x_2+x_4)+x_1x_2x_4\right) \\
\underline{x_5-x_2} & (x_5-x_2)(x_5+x_2+x_1) & (x_5-x_2)(x_5^2+x_2^2+x_1^2+x_1x_2+x_1x_5+x_2x_5) & (x_5-x_2)\left((x_5^2+x_2^2+x_1^2)(x_1+x_2+x_5)+x_1x_2x_5\right) \\
\underline{x_6-x_2} & (x_6-x_2)(x_6+x_2+x_1) & (x_6-x_2)(x_6^2+x_2^2+x_1^2+x_1x_2+x_1x_6+x_2x_6) & (x_6-x_2)\left((x_6^2+x_2^2+x_1^2)(x_1+x_2+x_6)+x_1x_2x_6\right)
\end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & (x_2 - x_1)(x_3 - x_1)(x_4 - x_1)(x_5 - x_1)(x_6 - x_1) \\
& \qquad \times (x_3 - x_2)(x_4 - x_2)(x_5 - x_2)(x_6 - x_2)
\end{aligned}
$$

$$
\times \det \begin{pmatrix}
1 & x_3 + x_2 + x_1 & x_3^2 + x_2^2 + x_1^2 + x_1x_2 + x_1x_3 + x_2x_3 & (x_3^2 + x_2^2 + x_1^2)(x_1 + x_2 + x_3) + x_1x_2x_3 \\
1 & x_4 + x_2 + x_1 & x_4^2 + x_2^2 + x_1^2 + x_1x_2 + x_1x_4 + x_2x_4 & (x_4^2 + x_2^2 + x_1^2)(x_1 + x_2 + x_4) + x_1x_2x_4 \\
1 & x_5 + x_2 + x_1 & x_5^2 + x_2^2 + x_1^2 + x_1x_2 + x_1x_5 + x_2x_5 & (x_5^2 + x_2^2 + x_1^2)(x_1 + x_2 + x_5) + x_1x_2x_5 \\
1 & x_6 + x_2 + x_1 & x_6^2 + x_2^2 + x_1^2 + x_1x_2 + x_1x_6 + x_2x_6 & (x_6^2 + x_2^2 + x_1^2)(x_1 + x_2 + x_6) + x_1x_2x_6
\end{pmatrix}
$$

$$
\begin{aligned}
= \quad & (x_2 - x_1)(x_3 - x_1)(x_4 - x_1)(x_5 - x_1)(x_6 - x_1) \\
& \qquad \times (x_3 - x_2)(x_4 - x_2)(x_5 - x_2)(x_6 - x_2)
\end{aligned}
$$

$$
\times \det \left( \begin{array}{c|ccc}
1 & x_3 + x_2 + x_1 & x_3^2 + x_2^2 + x_1^2 + x_1x_2 + x_1x_3 + x_2x_3 & (x_3^2 + x_2^2 + x_1^2)(x_1 + x_2 + x_3) + x_1x_2x_3 \\
0 & x_4 - x_3 & (x_4 - x_3)(x_4 + x_3 + x_2 + x_1) & (x_4 - x_3)\left(x_4^2 + x_3^2 + x_2^2 + x_1^2 + x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4\right) \\
0 & x_5 - x_3 & (x_5 - x_3)(x_5 + x_3 + x_2 + x_1) & (x_5 - x_3)\left(x_5^2 + x_3^2 + x_2^2 + x_1^2 + x_1x_2 + x_1x_3 + x_1x_5 + x_2x_3 + x_2x_5 + x_3x_5\right) \\
0 & x_6 - x_3 & (x_6 - x_3)(x_6 + x_3 + x_2 + x_1) & (x_6 - x_3)\left(x_6^2 + x_3^2 + x_2^2 + x_1^2 + x_1x_2 + x_1x_3 + x_1x_6 + x_2x_3 + x_2x_6 + x_3x_6\right)
\end{array} \right)
$$

$$
\begin{aligned}
= \quad & (x_2 - x_1)(x_3 - x_1)(x_4 - x_1)(x_5 - x_1)(x_6 - x_1) \\
& \qquad \times (x_3 - x_2)(x_4 - x_2)(x_5 - x_2)(x_6 - x_2)
\end{aligned}
$$

$$
\times \det \begin{pmatrix}
\underline{x_4 - x_3} & (x_4 - x_3)(x_4 + x_3 + x_2 + x_1) & (x_4 - x_3)\left(x_4^2 + x_3^2 + x_2^2 + x_1^2 + x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4\right) \\
\underline{x_5 - x_3} & (x_5 - x_3)(x_5 + x_3 + x_2 + x_1) & (x_5 - x_3)\left(x_5^2 + x_3^2 + x_2^2 + x_1^2 + x_1x_2 + x_1x_3 + x_1x_5 + x_2x_3 + x_2x_5 + x_3x_5\right) \\
\underline{x_6 - x_3} & (x_6 - x_3)(x_6 + x_3 + x_2 + x_1) & (x_6 - x_3)\left(x_6^2 + x_3^2 + x_2^2 + x_1^2 + x_1x_2 + x_1x_3 + x_1x_6 + x_2x_3 + x_2x_6 + x_3x_6\right)
\end{pmatrix}
$$

$$\begin{aligned}
= \quad & (x_2-x_1)(x_3-x_1)(x_4-x_1)(x_5-x_1)(x_6-x_1) \\
& \times(x_3-x_2)(x_4-x_2)(x_5-x_2)(x_6-x_2) \\
& \times(x_4-x_3)(x_5-x_3)(x_6-x_3)
\end{aligned}$$

$$\times \det \begin{pmatrix} 1 & x_4+x_3+x_2+x_1 & x_4^2+x_3^2+x_2^2+x_1^2+x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4 \\ 1 & x_5+x_3+x_2+x_1 & x_5^2+x_3^2+x_2^2+x_1^2+x_1x_2+x_1x_3+x_1x_5+x_2x_3+x_2x_5+x_3x_5 \\ 1 & x_6+x_3+x_2+x_1 & x_6^2+x_3^2+x_2^2+x_1^2+x_1x_2+x_1x_3+x_1x_6+x_2x_3+x_2x_6+x_3x_6 \end{pmatrix}$$

$$\begin{aligned}
= \quad & (x_2-x_1)(x_3-x_1)(x_4-x_1)(x_5-x_1)(x_6-x_1) \\
& \times(x_3-x_2)(x_4-x_2)(x_5-x_2)(x_6-x_2) \\
& \times(x_4-x_3)(x_5-x_3)(x_6-x_3)
\end{aligned}$$

$$\times \det \left(\begin{array}{c|cc} 1 & x_4+x_3+x_2+x_1 & x_4^2+x_3^2+x_2^2+x_1^2+x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4 \\ 0 & x_5-x_4 & (x_5-x_4)(x_5+x_4+x_3+x_2+x_1) \\ 0 & x_6-x_4 & (x_6-x_4)(x_6+x_4+x_3+x_2+x_1) \end{array}\right)$$

$$\begin{aligned}
= \quad & (x_2-x_1)(x_3-x_1)(x_4-x_1)(x_5-x_1)(x_6-x_1) \\
& \times(x_3-x_2)(x_4-x_2)(x_5-x_2)(x_6-x_2) \\
& \times(x_4-x_3)(x_5-x_3)(x_6-x_3)
\end{aligned}$$

$$\times \det \begin{pmatrix} \underline{x_5-x_4} & (x_5-x_4)(x_5+x_4+x_3+x_2+x_1) \\ \underline{x_6-x_4} & (x_6-x_4)(x_6+x_4+x_3+x_2+x_1) \end{pmatrix}$$

$$\begin{aligned}
= \quad & (x_2-x_1)(x_3-x_1)(x_4-x_1)(x_5-x_1)(x_6-x_1) \times \det \begin{pmatrix} 1 & x_5+x_4+x_3+x_2+x_1 \\ 1 & x_6+x_4+x_3+x_2+x_1 \end{pmatrix} \\
& \times(x_3-x_2)(x_4-x_2)(x_5-x_2)(x_6-x_2) \\
& \times(x_4-x_3)(x_5-x_3)(x_6-x_3) \\
& \times(x_5-x_4)(x_6-x_4)
\end{aligned}$$

$$\begin{aligned}
= \quad & (x_2-x_1)(x_3-x_1)(x_4-x_1)(x_5-x_1)(x_6-x_1) \times \det \begin{pmatrix} 1 & x_5+x_4+x_3+x_2+x_1 \\ 0 & x_6-x_5 \end{pmatrix} \\
& \times(x_3-x_2)(x_4-x_2)(x_5-x_2)(x_6-x_2) \\
& \times(x_4-x_3)(x_5-x_3)(x_6-x_3) \\
& \times(x_5-x_4)(x_6-x_4)
\end{aligned}$$

$$\begin{aligned}
= \quad & (x_2-x_1)(x_3-x_1)(x_4-x_1)(x_5-x_1)(x_6-x_1) \\
& \times(x_3-x_2)(x_4-x_2)(x_5-x_2)(x_6-x_2) \\
& \times(x_4-x_3)(x_5-x_3)(x_6-x_3) \\
& \times(x_5-x_4)(x_6-x_4) \\
& \times(x_6-x_5). \quad \blacksquare
\end{aligned}$$

(2) 行列式の交代多重線形性を用いて, 複素数を要素とする次の行列の行列式を求めよ.

**[(2-2) の解答例]**

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix} = \underline{(-1)^2} \cdot \det\left(\begin{array}{cc|cc} a_{13} & a_{14} & a_{11} & a_{12} \\ a_{23} & a_{24} & a_{21} & a_{22} \\ \hline 0 & 0 & a_{31} & a_{32} \\ 0 & 0 & a_{41} & a_{42} \end{array}\right) \quad \left(\begin{array}{l}\text{第 1 列と第 3 列} \\ \text{を入れ替え,} \\ \text{第 2 列と第 4 列} \\ \text{を入れ替える.}\end{array}\right)$$

$$= \det\begin{pmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{pmatrix} \cdot \det\begin{pmatrix} a_{31} & a_{32} \\ a_{41} & a_{42} \end{pmatrix}$$

$$= (a_{13}a_{24} - a_{14}a_{23})(a_{31}a_{42} - a_{32}a_{41}). \quad \blacksquare$$

**[(2-3) の解答例]**

$$\det\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & 0 & 0 \end{pmatrix} = \underline{(-1)} \cdot \det\left(\begin{array}{cc|cc} a_{11} & a_{12} & 0 & 0 \\ a_{41} & a_{42} & 0 & 0 \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{array}\right) \quad \left(\begin{array}{l}\text{第 2 行と第 4 行} \\ \text{を入れ替える.}\end{array}\right)$$

$$= (-1) \cdot \det\begin{pmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{pmatrix} \cdot \det\begin{pmatrix} a_{33} & a_{34} \\ a_{23} & a_{24} \end{pmatrix}$$

$$= -(a_{11}a_{42} - a_{12}a_{41})(a_{33}a_{24} - a_{34}a_{23}). \quad \blacksquare$$

**[(2-5) の解答例]**

$$\det\begin{pmatrix} 0 & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & 0 \end{pmatrix} = \underline{(-1)} \cdot \det\begin{pmatrix} a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & a_{12} & a_{13} & 0 \\ 0 & a_{42} & a_{43} & 0 \end{pmatrix} \quad \left(\begin{array}{l}\text{第 1 行と第 3 行} \\ \text{を入れ替える.}\end{array}\right)$$

$$= \underline{(-1)} \cdot (-1) \cdot \det\left(\begin{array}{cc|cc} a_{31} & a_{34} & a_{33} & a_{34} \\ a_{21} & a_{24} & a_{23} & a_{24} \\ \hline 0 & 0 & a_{13} & a_{12} \\ 0 & 0 & a_{43} & a_{42} \end{array}\right) \quad \left(\text{第 2 列と第 4 列を入れ替える.}\right)$$

$$= \det\begin{pmatrix} a_{31} & a_{34} \\ a_{21} & a_{24} \end{pmatrix} \cdot \det\begin{pmatrix} a_{13} & a_{12} \\ a_{43} & a_{42} \end{pmatrix}$$

$$= (a_{31}a_{24} - a_{34}a_{21})(a_{13}a_{42} - a_{12}a_{43}). \quad \blacksquare$$

幸山直人<br>2004 年 10 月 30 日