

2 行列式論

2.12.1 正則性の判定

C 次の行列の正則性を判定し, 正則ならば逆行列を余因子を用いて求めよ.

[(2) の解答例] 正則性を判別するために $\det A$ を計算する ($\det A \neq 0 \Leftrightarrow$ 正則).

$$\det A = \left(\frac{1}{2}\right)^4 \cdot \det\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \quad \left(\begin{array}{l} \text{第 1 列, 第 2 列, 第 3 列, 第 4 列を} \\ \left(\frac{1}{2}\right) \text{でそれぞれくくる.} \end{array}\right)$$

$$= \left(\frac{1}{2}\right)^4 \cdot \det\begin{pmatrix} 2 & 0 & 2 & 0 \\ 2 & 0 & -2 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \quad \left(\begin{array}{l} \text{第 1 行に第 3 行を足し,} \\ \text{第 2 行に第 4 行を足す.} \end{array}\right)$$

$$= \left(\frac{1}{2}\right)^4 \cdot 2^2 \cdot \det\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \quad \left(\begin{array}{l} \text{第 1 行を 2 でくくり,} \\ \text{第 2 行を 2 でくくる.} \end{array}\right)$$

$$= \left(\frac{1}{2}\right)^2 \cdot (-1) \cdot \det\left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ \hline 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \end{array}\right) \quad \left(\text{第 2 列と第 3 列を交換する.}\right)$$

$$= -\left(\frac{1}{2}\right)^2 \cdot \det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \det\begin{pmatrix} -1 & -1 \\ -i & i \end{pmatrix}$$

$$= -\left(\frac{1}{2}\right)^2 \cdot (-2) \cdot (-2i) = -i.$$

$\det A \neq 0$ より, 行列 A は正則で逆行列が存在する. 従って, 各余因子を求め, 逆行列を求める.

$$
\begin{aligned}
\Delta_{11} &= (-1)^{1+1}\cdot\left(\frac{1}{2}\right)^3\cdot\det\begin{pmatrix} i & -1 & -i \\ -1 & 1 & -1 \\ -i & -1 & i \end{pmatrix} \quad \left(\begin{array}{c}\text{第 1 列, 第 2 列, 第 3 列を} \\ \left(\frac{1}{2}\right)\text{ でそれぞれくくる.}\end{array}\right) \\
&= \left(\frac{1}{2}\right)^3\cdot\det\left(\begin{array}{ccc} 0 & -2 & 0 \\ \hline -1 & 1 & -1 \\ -i & -1 & i \end{array}\right) \quad \left(\text{第 1 行に第 3 行を足す.}\right) \\
&= \left(\frac{1}{2}\right)^3\cdot(-1)^{1+2}\cdot(-2)\cdot\det\begin{pmatrix} -1 & -1 \\ -i & i \end{pmatrix} \quad \left(\text{第 1 行に関してラプラス展開.}\right) \\
&= \left(\frac{1}{2}\right)^2\cdot(-2i) \quad = \quad -\frac{i}{2}.
\end{aligned}
$$

同様に,

$$
\begin{array}{llll}
\Delta_{11} = -\dfrac{i}{2}, & \Delta_{12} = -\dfrac{i}{2}, & \Delta_{13} = \dfrac{i}{2}, & \Delta_{14} = -\dfrac{i}{2}, \\[2ex]
\Delta_{21} = -\dfrac{i}{2}, & \Delta_{22} = -\dfrac{1}{2}, & \Delta_{23} = \dfrac{i}{2}, & \Delta_{24} = \dfrac{1}{2}, \\[2ex]
\Delta_{31} = -\dfrac{i}{2}, & \Delta_{32} = \dfrac{i}{2}, & \Delta_{33} = -\dfrac{i}{2}, & \Delta_{34} = \dfrac{i}{2}, \\[2ex]
\Delta_{41} = -\dfrac{i}{2}, & \Delta_{42} = \dfrac{1}{2}, & \Delta_{43} = \dfrac{i}{2}, & \Delta_{44} = -\dfrac{1}{2}.
\end{array}
$$

従って, 行列 A の逆行列 A^{-1} は

$$
\begin{aligned}
A^{-1} &= \frac{1}{\det A}\mathrm{cof}(A) = \frac{1}{\det A}\begin{pmatrix} \Delta_{11} & \Delta_{21} & \Delta_{31} & \Delta_{41} \\ \Delta_{12} & \Delta_{22} & \Delta_{32} & \Delta_{42} \\ \Delta_{13} & \Delta_{23} & \Delta_{33} & \Delta_{43} \\ \Delta_{14} & \Delta_{24} & \Delta_{34} & \Delta_{44} \end{pmatrix} \\
&= \frac{1}{-i}\cdot\begin{pmatrix} -\dfrac{i}{2} & -\dfrac{i}{2} & -\dfrac{i}{2} & -\dfrac{i}{2} \\[2ex] -\dfrac{i}{2} & -\dfrac{1}{2} & \dfrac{i}{2} & \dfrac{1}{2} \\[2ex] -\dfrac{i}{2} & \dfrac{i}{2} & -\dfrac{i}{2} & \dfrac{i}{2} \\[2ex] -\dfrac{i}{2} & \dfrac{1}{2} & \dfrac{i}{2} & -\dfrac{1}{2} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{2} & \dfrac{1}{2} & \dfrac{1}{2} & \dfrac{1}{2} \\[2ex] \dfrac{1}{2} & -\dfrac{i}{2} & -\dfrac{1}{2} & \dfrac{i}{2} \\[2ex] \dfrac{1}{2} & -\dfrac{1}{2} & \dfrac{1}{2} & -\dfrac{1}{2} \\[2ex] \dfrac{1}{2} & \dfrac{i}{2} & -\dfrac{1}{2} & -\dfrac{i}{2} \end{pmatrix}.
\end{aligned}
$$

■

[(4) の解答例] 正則性を判別するために $\det A$ を計算する ($\det A \neq 0 \Leftrightarrow$ 正則).

$$\det A = \left(\frac{1}{5}\right)^2 \cdot \det \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{4} & 1 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 1 \\ 1 & 1 & 1 & 1 & 5 \end{pmatrix} \quad \left(\begin{array}{l} \text{第 5 列を} \left(\frac{1}{5}\right) \text{でくくり,} \\ \text{第 5 行を} \left(\frac{1}{5}\right) \text{でくくる.} \end{array}\right)$$

$$= \left(\frac{1}{5}\right)^2 \cdot \left(\frac{1}{4}\right)^2 \cdot \det \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{3} & 1 & 1 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 1 & 1 \\ 1 & 1 & 1 & 4 & 4 \\ 1 & 1 & 1 & 4 & 5 \end{pmatrix} \quad \left(\begin{array}{l} \text{第 4 列を} \left(\frac{1}{4}\right) \text{でくくり,} \\ \text{第 4 行を} \left(\frac{1}{4}\right) \text{でくくる.} \end{array}\right)$$

$$= \left(\frac{1}{5} \cdot \frac{1}{4}\right)^2 \cdot \left(\frac{1}{3}\right)^2 \cdot \det \begin{pmatrix} 1 & \frac{1}{2} & 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 & 1 & 1 \\ 1 & 1 & 3 & 3 & 3 \\ 1 & 1 & 3 & 4 & 4 \\ 1 & 1 & 3 & 4 & 5 \end{pmatrix} \quad \left(\begin{array}{l} \text{第 3 列を} \left(\frac{1}{3}\right) \text{でくくり,} \\ \text{第 3 行を} \left(\frac{1}{3}\right) \text{でくくる.} \end{array}\right)$$

$$= \left(\frac{1}{5} \cdot \frac{1}{4} \cdot \frac{1}{3}\right)^2 \cdot \left(\frac{1}{2}\right)^2 \cdot \det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \quad \left(\begin{array}{l} \text{第 2 列を} \left(\frac{1}{2}\right) \text{でくくり,} \\ \text{第 2 行を} \left(\frac{1}{2}\right) \text{でくくる.} \end{array}\right)$$

$$= \left(\frac{1}{5} \cdot \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{2}\right)^2 \cdot \det \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \left(\text{第 5 行から第 4 行を引く.}\right)$$

$$= \left(\frac{1}{5}\cdot\frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2}\right)^2\cdot\det\begin{pmatrix}1&1&1&1&1\\1&2&2&2&2\\1&2&3&3&3\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}\quad\Big(\text{第 4 行から第 3 行を引く.}\Big)$$

$$= \left(\frac{1}{5}\cdot\frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2}\right)^2\cdot\det\begin{pmatrix}1&1&1&1&1\\1&2&2&2&2\\0&0&1&1&1\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}\quad\Big(\text{第 3 行から第 2 行を引く.}\Big)$$

$$= \left(\frac{1}{5}\cdot\frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2}\right)^2\cdot\det\begin{pmatrix}1&1&1&1&1\\0&1&1&1&1\\0&0&1&1&1\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix}\quad\Big(\text{第 2 行から第 1 行を引く.}\Big)$$

$$= \left(\frac{1}{5}\cdot\frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2}\right)^2\cdot 1 \quad = \quad \frac{1}{14400}.$$

$\det A \neq 0$ より, 行列 A は正則で逆行列が存在する. 従って, 各余因子を求め, 逆行列を求める.

$$\Delta_{11} = (-1)^{1+1}\cdot\left(\frac{1}{5}\right)^2\cdot\det\begin{pmatrix}\frac{1}{2}&\frac{1}{3}&\frac{1}{4}&1\\\frac{1}{3}&\frac{1}{3}&\frac{1}{4}&1\\\frac{1}{4}&\frac{1}{4}&\frac{1}{4}&1\\1&1&1&5\end{pmatrix}\quad\left(\begin{array}{l}\text{第 4 列を }\left(\frac{1}{5}\right)\text{ でくくり,}\\\text{第 4 行を }\left(\frac{1}{5}\right)\text{ でくくる.}\end{array}\right)$$

$$= \left(\frac{1}{5}\right)^2\cdot\left(\frac{1}{4}\right)^2\cdot\det\begin{pmatrix}\frac{1}{2}&\frac{1}{3}&1&1\\\frac{1}{3}&\frac{1}{3}&1&1\\1&1&4&4\\1&1&4&5\end{pmatrix}\quad\left(\begin{array}{l}\text{第 3 列を }\left(\frac{1}{4}\right)\text{ でくくり,}\\\text{第 3 行を }\left(\frac{1}{4}\right)\text{ でくくる.}\end{array}\right)$$

$$= \left(\frac{1}{5}\cdot\frac{1}{4}\right)^2\cdot\left(\frac{1}{3}\right)^2\cdot\det\begin{pmatrix}\frac{1}{2}&1&1&1\\1&3&3&3\\1&3&4&4\\1&3&4&5\end{pmatrix}\quad\left(\begin{array}{l}\text{第 2 列を }\left(\frac{1}{3}\right)\text{ でくくり,}\\\text{第 2 行を }\left(\frac{1}{3}\right)\text{ でくくる.}\end{array}\right)$$

$$= \left(\frac{1}{5}\cdot\frac{1}{4}\cdot\frac{1}{3}\right)^2\cdot\left(\frac{1}{2}\right)^2\cdot\det\begin{pmatrix}2&2&2&2\\2&3&3&3\\2&3&4&4\\2&3&4&5\end{pmatrix} \quad \left(\begin{array}{l}\text{第 1 列を}\left(\frac{1}{2}\right)\text{でくくり,}\\ \\ \text{第 1 行を}\left(\frac{1}{2}\right)\text{でくくる.}\end{array}\right)$$

$$= \left(\frac{1}{5}\cdot\frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2}\right)^2\cdot\det\begin{pmatrix}2&2&2&2\\2&3&3&3\\2&3&4&4\\0&0&0&1\end{pmatrix} \quad \left(\text{第 4 行から第 3 行を引く.}\right)$$

$$= \left(\frac{1}{5}\cdot\frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2}\right)^2\cdot\det\begin{pmatrix}2&2&2&2\\2&3&3&3\\0&0&1&1\\0&0&0&1\end{pmatrix} \quad \left(\text{第 3 行から第 2 行を引く.}\right)$$

$$= \left(\frac{1}{5}\cdot\frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2}\right)^2\cdot\det\begin{pmatrix}2&2&2&2\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{pmatrix} \quad \left(\text{第 2 行から第 1 行を引く.}\right)$$

$$= \left(\frac{1}{5}\cdot\frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2}\right)^2\cdot 2 \quad = \quad \frac{1}{7200}.$$

同様に,

$$\begin{array}{lllll}
\Delta_{11} = \frac{1}{7200}, & \Delta_{12} = -\frac{1}{7200}, & \Delta_{13} = 0, & \Delta_{14} = 0, & \Delta_{15} = 0, \\
\Delta_{21} = -\frac{1}{7200}, & \Delta_{22} = \frac{1}{1800}, & \Delta_{23} = -\frac{1}{2400}, & \Delta_{24} = 0, & \Delta_{25} = 0, \\
\Delta_{31} = 0, & \Delta_{32} = -\frac{1}{2400}, & \Delta_{33} = \frac{1}{800}, & \Delta_{34} = -\frac{1}{1200}, & \Delta_{35} = 0, \\
\Delta_{41} = 0, & \Delta_{42} = 0, & \Delta_{43} = -\frac{1}{1200}, & \Delta_{44} = \frac{1}{450}, & \Delta_{45} = -\frac{1}{720}, \\
\Delta_{51} = 0, & \Delta_{52} = 0, & \Delta_{53} = 0, & \Delta_{54} = -\frac{1}{720}, & \Delta_{55} = \frac{1}{576}.
\end{array}$$

従って, 行列 A の逆行列 A^{-1} は

$$
\begin{aligned}
A^{-1} &= \frac{1}{\det A}\mathrm{cof}(A) = \frac{1}{\det A}\begin{pmatrix} \Delta_{11} & \Delta_{21} & \Delta_{31} & \Delta_{41} & \Delta_{51} \\ \Delta_{12} & \Delta_{22} & \Delta_{32} & \Delta_{42} & \Delta_{52} \\ \Delta_{13} & \Delta_{23} & \Delta_{33} & \Delta_{43} & \Delta_{53} \\ \Delta_{14} & \Delta_{24} & \Delta_{34} & \Delta_{44} & \Delta_{54} \\ \Delta_{15} & \Delta_{25} & \Delta_{35} & \Delta_{45} & \Delta_{55} \end{pmatrix} \\
&= 14400 \cdot \begin{pmatrix} \frac{1}{7200} & -\frac{1}{7200} & 0 & 0 & 0 \\ -\frac{1}{7200} & \frac{1}{1800} & -\frac{1}{2400} & 0 & 0 \\ 0 & -\frac{1}{2400} & \frac{1}{800} & -\frac{1}{1200} & 0 \\ 0 & 0 & -\frac{1}{1200} & \frac{1}{450} & -\frac{1}{720} \\ 0 & 0 & 0 & -\frac{1}{720} & \frac{1}{576} \end{pmatrix} \\
&= \begin{pmatrix} 2 & -2 & 0 & 0 & 0 \\ -2 & 8 & -6 & 0 & 0 \\ 0 & -6 & 18 & -12 & 0 \\ 0 & 0 & -12 & 32 & -20 \\ 0 & 0 & 0 & -20 & 25 \end{pmatrix}.
\end{aligned}
$$

■

幸山直人

2004 年 11 月 6 日